Firm behavior with inventory

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Abstract

In this paper I describe the behavior of manufacturing firm that produces final output at a remote location and fulfills consumer demand through its retail units. The model is setup in infinite horizon continuous time. I argue that the price at any moment will depend on the price at the previous moment.

1 Introduction

I study a partial equilibrium model of inventory management in infinite horizon continuous time. In this model a manufacturing firm produces output at a remote location. Therefore, it must use its retail units to satisfy consumer demand. The motivation behind this work is to develop a simple model of inventory management that can be combined into general equilibrium models in a tractable manner. There are various version of inventory management models explored in economics and operation research literature, however, to the best of my knowledge, all of them are partial equilibrium models that are not straightforward to incorporate into general equilibrium framework.

It is argued that output price at any moment t depends upon the price at an earlier moment $t - dt$. The implication of this result is that it is a step toward explaining price stickiness in New-Keyneysian monetary economics models where it assumed to be exogenously given.

Relation to the literature

This work builds upon the existing literature on inventory management. Some of the early works in this area were done in Arrow et al. (1951), and Bellman et al. (1955). They derive optimal inventory policy under various conditions, however, assuming demand function to be exogenously given. Scarf (1960), and Iglehart (1963) explore the optimality of (s,S) inventory policy in a dynamic set up.

The (s,S) policy of inventory management is one of the widely studied model by economists. Iglehart (1963) study convergence of economies aggregate to frictionless counterpart. Khan and Thomas (2007) incorporate (s,S) into a business cycle model to explain cyclical variability of inventory investment.

The inventory management problem has also been extensively explored in operations research literature. This paper is close to continuous time models as described in Benkherouf and Gilding (2021a), Benkherouf and Gilding (2021b), and Alfares and Ghaithan (2016). The contribution of this paper is to study the inventory decision with non linear price dependent demand rate.

2 Model

The problem is setup in infinite horizon continuous time. The economic agents have perfect foresight and there is no uncertainty. There is a manufacturing firm that produces a single good as its output using labor as only input. The manufacturing activity takes place at a location different from the location of sale. So, in order to be able to sell its product the manufacturer relies on its retail unit that maintains an inventory of the good. The retail unit faces continuous consumer demand and orders inventory at discrete interval of time once the stock becomes too low. For each order it has to pay administrative cost, independent of the quantity ordered, on top of the per unit price.

2.1 Manufacturing unit

The manufacturing unit produces a single good as output using a decreasing returns to scale technology using only labor as input. The manufacturing unit has to supply whatever quantity the retail unit orders at discrete time interval. Suppose at time t_i the retail unit orders ξ_i quantity of goods then the manufacturing firm's optimization problem can be stated as follows.

$$\pi_m = \max_{y_t,\ell_t,c_{t_i}} \left\{ \int_0^\infty e^{-\rho t}(-w_t\ell_t)dt + \sum_{i=1}^{\infty} c_{t_i}\xi_i e^{-\rho t_i} \right\}$$
$$\text{subject to} \quad y_t = A_t\ell_t^\alpha.$$

where $\alpha \in (0,1)$ [1]. Market clearing of the output at discrete interval requires

$$\xi_i = \int_{t_{i-1}}^{t_i} y_t dt; \quad \forall i = 1, \dots,$$

and $t_0 = 0$.

Assumption 1. *At each intervention moment (t_i), the manufacturing unit breaks even in output market.*

[1] Since this is a partial equilibrium analysis, therefore $\alpha = 1$, constant returns to scale technology, will result in firm demanding infinite amount to labor to produce infinite amount of output. In general equilibrium model, this situation is prevented from arising due to supply side constraint.

Claim 1. *Under Assumption 1, the solution to the firm's optimization problem is characterized by the following equations:*

$$\alpha c_{t_i} e^{-\rho t_i} A_t \ell_t^{\alpha-1} = w_t e^{-\rho t} \tag{1}$$

2.2 Retail unit

The retail unit begins with some positive stock of goods (x_0) and faces the consumer demand which is assumed to be deterministic. If the stock is too low then it makes an order to the manufacturing unit which costs k irrespective of the size of the order, in addition it also pays a per unit cost c_t. Once the order is fulfilled the retailer also incurs a storage cost $(f(x))$ which is increasing in the size of the inventory. The retailers optimization problem can be stated as follows.

$$\pi_r(x_0) = \max_{p_t, \{t_i, \xi_i\}_{i=1}^{\infty}} \left\{ \int_{t=0}^{\infty} e^{-\rho t}[-p_t \dot{x}_t - f(x_t)]dt - \sum_{i=1}^{\infty}(k + c_{t_i}\xi_i)e^{-\rho t_i} \right\}$$

The rate of decrease in inventory will be equal to demand rate. The relationship is given by

$$\dot{x}_t = -\kappa p_t^{\frac{-1}{1-\epsilon}},$$

where $0 < \epsilon < 1$.

Now consider the interval $[t_i, t_{i+1})$, the retail unit orders inventory at t_i which is sold by time t_{i+1}. The firm's reward in this interval will be

$$\int_{t_i}^{t_{i+1}} e^{-\rho(t-t_i)}[-p_t \dot{x}_t - f(x_t)]dt - \int_{t_{i-1}}^{t_i} c_{t_i} y_t dt - k \tag{2}$$

From Equation (2), observe that it is optimal for the retailer to set its price such that the entire stock of output is sold out immediately, so that the storage cost of inventory is minimized.

Claim 2. *The optimal strategy of the retailer is to set the price p_t such that entire* *inventory is sold out immediately.*

Proof. Given the rate of demand is $\kappa p_t^{-\frac{1}{1-\epsilon}}$, the revenue at any moment as a function of price is given by $\kappa p_t^{-\frac{\epsilon}{1-\epsilon}}$ which is a decreasing function of p_t.

Secondly, in general, the storage cost, $(f(x_t))$ will be an increasing function of the amount of inventory being stored.

Lastly, the cost of accumulating inventory at t_i, $\int_{t_{i-1}}^{t_i} c_{t_i} y_t dt + k$ has already been incurred at t_i and therefore it acts as a fixed quantity in the interval $[t_i, t_{i+1}]$.

Thus retailer optimization will imply that it sets price p_t low enough to sell all the inventory immediately, which simultaneously maximizes the revenue and reduces the storage cost. ∎

Since the inventory will be ordered each instant, so the ordering cost k will now be interpreted as ordering cost per unit time. In other words, the ordering cost paid by the retailer at t will be kdt. Hence the retail units optimization problem between period t and $t + dt$ can be written as

$$\int_t^{t+dt} e^{-\rho(\tau-t)}[-p_\tau \dot{x}_\tau - f(x_\tau)]d\tau - c_t\xi_t - kdt \tag{3}$$

$$= [-p_t\dot{x}_t - f(x_t)]dt - c_t\xi_t - kdt$$

The above equation is further simplified in the following manner. The rate of decrease in inventory is

$$\dot{x}_t = -\kappa p_t^{-\frac{1}{1-\epsilon}}$$

$$\Rightarrow \int_{\xi_t}^{x_t} dx_\tau = -\kappa \int_t^{t+dt} p_\tau^{-\frac{1}{1-\epsilon}} d\tau$$

Recall that at every moment t, the retail unit orders inventory ξ_t which is sold out by $t + dt$, before it begins the new cycle. Hence,

$$\xi_t = \kappa p_t^{-\frac{1}{1-\epsilon}} dt$$

Assumption 2. *The functional form of $f(x)$ is given by $f(x) = \sigma x$, where σ is a constant parameter of the model.*

Now the retailers optimization problem at t becomes

$$\left\{\kappa p_t^{-\frac{\epsilon}{1-\epsilon}} - (\sigma + c_t)\kappa p_t^{-\frac{1}{1-\epsilon}}\right\} dt - kdt.$$

Claim 3. *The profit maximizing price is $p_t = \dfrac{\sigma + c_t}{\epsilon}$. And the quantity of inventory ordered is $\kappa \left(\dfrac{\sigma + c_t}{\epsilon}\right)^{-\frac{1}{1-\epsilon}}$.*

From Equation (1), the supply rate is $A_{t-dt}\left(\dfrac{\alpha A_{t-dt} c_t}{w_{t-dt}}\right)^{\frac{\alpha}{1-\alpha}}$. The per unit ordering cost c_t can be obtained by equating the demand rate and supply rate. That is, c_t is determined by

$$\kappa \left(\frac{\sigma + c_t}{\epsilon}\right)^{-\frac{1}{1-\epsilon}} = A_{t-dt}\left(\frac{\alpha A_{t-dt} c_t}{w_{t-dt}}\right)^{\frac{\alpha}{1-\alpha}}. \tag{4}$$

In the special case of no storage cost, $\sigma = 0$, the expression for c_t can be determined analytically,

$$c_t^{\frac{\alpha}{1-\alpha}+\frac{1}{1-\epsilon}} = \left(\frac{\kappa \epsilon^{\frac{1}{1-\epsilon}}}{\alpha^{\frac{\alpha}{1-\alpha}}}\right) \cdot \left(\frac{w_{t-dt}^{\alpha}}{A_{t-dt}}\right)^{\frac{1}{1-\alpha}}.$$

Through Equation (4), c_t is a function of prices in the previous instant w_{t-dt}, and therefore through its dependence on c_t, p_t is a function of previous period price.

A model with no inventory

For comparison, consider a model of a monopolist firm producing the output with same technology, $y_t = A_t ell_t^{\alpha}$, and facing the same demand rate $y_t = \kappa p_t^{-\frac{1}{1-\epsilon}}$, where l_t is the labor input employed, A_t is the productivity, and p_t is the price of the final output.

The profit maximization of this firm can be stated as follows:

$$\max_{p_t} p_t y_t - w_t \ell_t,$$

$$\text{s.t.} \quad y_t = A_t \ell_t^{\alpha},$$

$$y_t = \kappa p_t^{-\frac{1}{1-\epsilon}}.$$

For this model, it can be shown that the firm will optimally set its price to be

$$p_t = \left[\frac{\kappa^{\frac{1-\alpha}{\alpha}}}{\alpha\epsilon} \cdot \frac{w_t}{A_t^{\frac{1}{\alpha}}} \right]^{\frac{1}{1+\frac{1-\alpha}{\alpha(1-\epsilon)}}},$$

which is independent of prices in the previous instant.

Steady state

Let us take the case where $w_t = w$, and $A_t = A$ for all $t \geq 0$, and $\sigma = 0$. So,

$$c = c_t = \left(\frac{\kappa \epsilon^{\frac{1}{1-\epsilon}}}{\alpha^{\frac{\alpha}{1-\alpha}}} \right)^{\frac{(1-\alpha)(1-\epsilon)}{1-\alpha\epsilon}} \cdot \left(\frac{w^{\alpha}}{A} \right)^{\frac{1-\epsilon}{1-\alpha\epsilon}} \tag{5a}$$

$$p = p_t = \frac{c}{\epsilon}. \tag{5b}$$

Equation (5) is a steady state results for the special case.

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